

EINSTEIN FOUR-MANIFOLDS WITH SKEW TORSION

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ABSTRACT. We develop a notion of Einstein manifolds with skew torsion on compact, orientable Riemannian manifolds of dimension four. We prove an analogue of the Hitchin-Thorpe inequality and study the case of equality. We use the link with self-duality to study the moduli space of 1-instantons on S^4 for a family of metrics defined by Bonneau.

1. INTRODUCTION

Torsion, and in particular skew torsion, has been a topic of interest to both mathematicians and physicists in recent decades. The first attempts to introduce torsion in general relativity go back to the 1920's with the work of É. Cartan, [9]. More recently, torsion makes its appearance in string theory, where the basic model of type II string theory consists of a Riemannian manifold, a connection with skew torsion, a spinorial field and a dilaton function, [1].

From the mathematical point of view, skew torsion has played a significant role in the work of Bismut and his local index theorem for non-Kähler manifolds, [5]. Bismut showed that for any Hermitian manifold, there is a unique connection with skew torsion which preserves both the metric and the complex structure. Nowadays these connections are known as Bismut connections and a Hermitian manifold equipped with such a connection is often referred to as a KT (Kähler with torsion) manifold. Also, skew torsion is an important feature in Hitchin and Gualtieri's generalized geometry, [17, 15] where there are natural connections with skew torsion, the exterior derivative of the B-field. In particular, two Bismut connections appear in the characterization of a generalized Kähler structure.

In this article, we propose a notion of Einstein manifold with skew torsion for a four-manifold. Four dimensions is of particular interest because of the phenomenon of self-duality. We define our notion of Einstein by making use of the decomposition of the curvature operator in terms of the action of $SO(4)$ and making the analogy with the standard Riemannian situation. Motivated by the earlier work of Hitchin and Thorpe, [16, 22], we show that an Einstein manifold with skew torsion satisfies a topological constraint, an inequality involving the Euler characteristic and the signature of the manifold: $2\chi \geq 3|\tau|$. Manifolds of type $S^1 \times S^3$ are well known to satisfy the inequality but they do not carry an Einstein metric. In fact they have a natural structure of flat manifold with skew torsion and we also prove that these are the only non-Einstein manifolds that satisfy the equality $2\chi = 3|\tau|$.

Our definition of Einstein with skew torsion depends on the choice of orientation but, as we show, this choice is irrelevant in the compact world. This is seen by establishing a one-to-one correspondence between Einstein manifolds with skew torsion and Einstein-Weyl manifolds and making use of the Gauduchon gauge where the torsion is closed.

Date: 14 July 2011.

2000 Mathematics Subject Classification. Primary 53C07; Secondary 14D21, 53C25.

An interesting observation is that a connection which is Einstein with skew torsion induces a self-dual connection on the bundle of self-dual forms and, if the manifold is spin, on the bundle of positive half-spinors. In view of this, we use the link with Einstein-Weyl, to present an example of a one-parameter family of $U(2)$ -invariant Einstein metrics with closed skew torsion on the 4-sphere, the triple (S^4, ds^2, H) as defined by G. Bonneau, [6]. We investigate the moduli space of charge 1 instantons and prove that it is diffeomorphic to the space of instantons for a metric of constant sectional curvature. The connections with torsion $\pm H$ define two different instantons which we point out to generically define the line of gauge equivalence classes of $U(2)$ -invariant instantons in the moduli space.

2. METRIC CONNECTIONS WITH SKEW TORSION

Let (M, g) be a Riemannian manifold. Suppose that ∇ is a connection on the tangent bundle of M and let T be its $(1,2)$ torsion tensor. If we contract T with the metric we get a $(0,3)$ tensor which we will still call the torsion of ∇ . If T is a three-form then we say that ∇ is a connection with skew-symmetric torsion. Given any three-form H on M then there exists a unique metric connection with skew torsion H defined explicitly by

$$g(\nabla_X Y, Z) = g(\nabla_X^g Y, Z) + \frac{1}{2}H(X, Y, Z)$$

where ∇^g is the Levi-Civita connection.

Consider now the triple (M, g, H) and let ∇ be the connection with skew torsion H . If R^g is the Riemannian curvature tensor and R^∇ is the curvature tensor associated with ∇ , we can express R^∇ in terms of R^g and H as follows: for every four vector fields X, Y, Z, W , we have

$$(2.1) \quad \begin{aligned} R^\nabla(X, Y, Z, W) &= R^g(X, Y, Z, W) \\ &+ \frac{1}{4}g(H(X, W), H(Y, Z)) - \frac{1}{4}g(H(Y, W), H(X, Z)) \\ &- \frac{1}{2}(\nabla_X^g H)(Y, Z, W) + \frac{1}{2}(\nabla_Y^g H)(X, Z, W). \end{aligned}$$

The tensor R^∇ will not have the same symmetries as R^g , as can be expected. For example, the analogue of the Bianchi identity is given by

$$(2.2) \quad \begin{aligned} R^\nabla(X, Y, Z, W) + R^\nabla(Y, Z, X, W) + R^\nabla(Z, X, Y, W) = \\ -dH(X, Y, Z, W) - (\nabla_W^g H)(X, Y, Z) + \frac{1}{2} \sigma_{XYZ} g(H(X, Y), H(Z, W)), \end{aligned}$$

where σ_{XYZ} denotes the cyclic sum over X, Y, Z .

Also, if ∇^- is the metric connection with torsion $-H$, we have

$$(2.3) \quad R^\nabla(X, Y, Z, W) = R^{\nabla^-}(Z, W, X, Y) - \frac{1}{2} dH(X, Y, Z, W)$$

In particular if H is closed, we obtain

$$(2.4) \quad R^\nabla(X, Y, Z, W) = R^{\nabla^-}(Z, W, X, Y).$$

Suppose now that M is orientable. If $n = \dim M$, let $\{e_i\}_{i=1}^n$ denote a positively oriented orthonormal frame of TM and $\{e^i\}_{i=1}^n$ its dual frame. Also, let Ric^g be the

usual Ricci tensor and Ric^∇ be the Ricci tensor with respect to ∇ . Given any pair of vector fields X, Y we have

$$(2.5) \quad \text{Ric}^\nabla(X, Y) = \text{Ric}^g(X, Y) - \frac{1}{4} \sum_i g(H(X, e_i), H(Y, e_i)) - \frac{1}{2} d^* H(X, Y).$$

Notice that, unlike Ric^g , Ric^∇ has a non-vanishing anti-symmetric part. For the scalar curvature, the relation is

$$(2.6) \quad s^\nabla - s^g = -\frac{3}{2} \|H\|^2$$

These or similar identities are already available in the literature. The original proof of 2.3-2.4 was done in [5]. For the remaining ones, see [18].

3. DECOMPOSITION OF THE RIEMANN TENSOR

We now restrict our attention to manifolds of dimension four. We will also be assuming compactness and orientability. Recall that the bundle of two-forms splits as $\Lambda^2 = \Lambda_+ \oplus \Lambda_-$, where Λ_+ and Λ_- are the bundles of self-dual and antiself-dual forms, respectively.

Consider (M, g, H) and notice that in four dimensions, the star operator also allows us to see the torsion H as a one-form. Let $h = *H$ and call h the torsion 1-form. One of the features of h is that it provides us with a simple way of writing the expression for the Ricci tensor.

Proposition 3.1. *On a four-dimensional manifold, the Ricci tensor for a connection ∇ with skew torsion H can be written as*

$$\text{Ric}^\nabla = \text{Ric}^g - \frac{1}{2} \|h\|^2 g + \frac{1}{2} h \otimes h - \frac{1}{2} * dh,$$

where g is the metric tensor.

Proof — Direct computation using the orthonormal frame $\{e_i\}$. □

The curvature tensor R^∇ of a connection with skew torsion lives in $\Lambda^2 \otimes \Lambda^2$. Using the metric, we can see R^∇ as a map $\mathcal{R}^\nabla : \Lambda^2 \rightarrow \Lambda^2$, called the curvature operator, which is given by the prescription

$$g(\mathcal{R}^\nabla(X \wedge Y), Z \wedge W) = R^\nabla(X, Y, Z, W).$$

We are going to work out the decomposition of \mathcal{R}^∇ in terms of the splitting $\Lambda^2 = \Lambda_+ \oplus \Lambda_-$. First let us recall briefly what happens in the usual Riemannian situation. The symmetries of R^g mean that this is an element of $S^2 \Lambda^2$, which can be decomposed as follows, [3].

$$\mathcal{R}^g = \left(\begin{array}{c|c} W^+ + \frac{s}{12} \text{Id} & Z \\ \hline Z^t & W^- + \frac{s}{12} \text{Id} \end{array} \right)$$

where s is the scalar curvature, W^+ and W^- are the self dual and antiself-dual parts of the Weyl tensor, and Z is the trace-free part of the Ricci tensor, i.e., $Z = \text{Ric}^g - \frac{s}{4}g$.

For a connection with skew torsion the situation is slightly more complicated, since R^∇ has a non-vanishing part in $\Lambda^2(\Lambda^2)$.

Theorem 3.2. *For a metric connection with skew torsion ∇ , we can decompose the Riemann curvature map \mathcal{R}^∇ in terms of self-dual and antiself-dual blocks as:*

$$\mathcal{R}^\nabla = \left(\begin{array}{c|c} W^+ + \left(\frac{s^\nabla}{12} - \frac{*dH}{4} \right) \text{Id} + \frac{1}{4} (d^*H)_+ & Z^\nabla + S(\nabla *H) + \frac{*dH}{4}g \\ \hline \left(Z^\nabla - S(\nabla *H) + \frac{*dH}{4}g \right)^\dagger & W^- + \left(\frac{s^\nabla}{12} + \frac{*dH}{4} \right) \text{Id} - \frac{1}{4} (d^*H)_- \end{array} \right)$$

where S denotes the symmetrization of a tensor and \dagger the transpose of a matrix, Z^∇ is the symmetric trace-free part of Ric^∇ , and $(d^*H)_+$ and $(d^*H)_-$ are the self-dual and antiself-dual part of d^*H , respectively.

Proof — We start with the upper left block and call it A . The best way to see what the entries are is to do an example. We will be using the convention R_{ijkl} for $R(e_i, e_j, e_k, e_l)$. Write $R = R^g + \bar{R}$ and recall equation 2.1. Take the first diagonal entry, this is given by

$$A_{11} = \frac{1}{2} (R_{1212} + R_{1234} + R_{3412} + R_{3434}).$$

We only need to worry about the \bar{R} component. We can easily see that

$$\bar{R}_{1212} + \bar{R}_{3424} = -\frac{1}{4} ((H_{12}^r)^2 + (H_{34}^r)^2) = -\frac{1}{4} \|H\|^2$$

and that

$$\bar{R}_{1234} + \bar{R}_{3412} = \frac{1}{2} (H_{14}^r H_{23}^r - H_{13}^r H_{24}^r) - \frac{1}{2} (dH)_{1234}$$

and since, $H_{ij}^r H_{kl}^r$ vanishes if i, j, k, l are all distinct, we get

$$\bar{A}_{11} = -\frac{\|H\|^2}{8} - \frac{*dH}{4}$$

and the same holds for the other diagonal entries. Consider the off-diagonal entries now. Taking A_{12} , for instance, we get that

$$\bar{A}_{12} = -\frac{1}{8} (H_{12}^r H_{13}^r - H_{12}^r H_{24}^r + H_{34}^r H_{13}^r - H_{34}^r H_{24}^r) + \frac{1}{4} ((d^*H)_{14} + (d^*H)_{23}).$$

The quadratic part of the expression vanishes, so we obtain

$$\bar{A}_{12} = \frac{1}{4} ((d^*H)_{14} + (d^*H)_{23})$$

and the other off-diagonal entries are analogous. Then, clearly, we have

$$A = W^+ + \left(\frac{s}{12} - \frac{\|H\|^2}{8} - \frac{*dH}{4} \right) \text{Id} + \frac{(d^*H)_+}{4}$$

and since $s^\nabla = s^g - \frac{3}{2}\|H\|^2$, by equation 2.6, we get the desired expression for A . If D is the lower right block then the arguments are perfectly similar to the ones for A . Consider now the upper right block, B . Let us start with the diagonal entries. Take

$$\bar{B}_{11} = \frac{1}{2} (\bar{R}_{1212} + \bar{R}_{1234} - \bar{R}_{3412} - \bar{R}_{3434}).$$

We have

$$\begin{aligned} \bar{R}_{1212} - \bar{R}_{3434} &= -\frac{1}{4} ((H_{123})^2 + (H_{124})^2 - (H_{134})^2 - (H_{234})^2) \\ \bar{R}_{1234} - \bar{R}_{3412} &= -\frac{1}{2} ((\nabla_1^g H)_{234} - (\nabla_2^g H)_{134} - (\nabla_3^g H)_{412} + (\nabla_4^g H)_{312}) \end{aligned}$$

and we will now write this in terms of $h = *H$, since it makes the calculations easier. We then have that

$$\bar{B}_{11} = \frac{1}{8} ((h_1)^2 + (h_2)^2 - (h_3)^2 - (h_4)^2) + \frac{1}{2} ((\nabla_1^g h)_1 + (\nabla_2^g h)_2 - (\nabla_3^g h)_3 - (\nabla_4^g h)_4)$$

and analogously for \bar{B}_{22} and \bar{B}_{33} . Consider now

$$\bar{B}_{12} = \frac{1}{2} (\bar{R}_{1312} + \bar{R}_{1334} + \bar{R}_{2412} + \bar{R}_{2434})$$

and we see that

$$\bar{B}_{12} = \frac{1}{4} (H_{123}H_{234} - H_{124}H_{134} + (\nabla_1^g H)_{132} - (\nabla_3^g H)_{314} - (\nabla_2^g H)_{241} + (\nabla_4^g H)_{423})$$

and, rewriting in terms of h , we get

$$\bar{B}_{12} = \frac{1}{4} (h_2h_3 - h_1h_4 + (\nabla_2^g h)_3 + (\nabla_3^g h)_2 - (\nabla_1^g h)_4 - (\nabla_4^g h)_1)$$

and we have similar results for the other entries. We wish to express B in terms of symmetric trace-free 2-tensors, so we need to choose the right isomorphism between $\Lambda_+ \otimes \Lambda_-$ and S_0^2 . This can be found in [3], is called the Ricci contraction and works as follows: if we consider 2-forms as matrices then $-\varphi$ is given by standard matrix multiplication. For example, the form $e^1 \wedge e^2$ corresponds to the 4×4 matrix M such that $M_{21} = 1, M_{12} = -1$ and $M_{ij} = 0$ elsewhere.

We can now conclude that

$$\bar{B} = \frac{1}{2} \left(h \otimes h - \frac{1}{4} \|h\|^2 g \right) + S(\nabla^g h) + \frac{d^*h}{4} g$$

where S denotes the symmetrization of the tensor. Observe the following two lemmas, which can be proved by simple local calculations.

Lemma 3.3. *If ∇ is the metric connection with skew torsion H , and $h = *H$, then $\nabla h = \nabla^g h$.*

Lemma 3.4. *The trace-free symmetric part of the Ricci tensor Ric^∇ , denoted by Z^∇ , is given by:*

$$Z^\nabla = Z^g + \frac{1}{2} h \otimes h - \frac{1}{8} \|h\|^2 g.$$

Finally, we get that

$$B = Z^\nabla + S(\nabla * H) + \frac{*dH}{4} g.$$

If C is the remaining block, by means of equation 2.3 and noticing that we always have two repeated indices, C is the transpose of B when replacing H by $-H$. \square

4. EINSTEIN METRICS WITH SKEW TORSION

The above decomposition of the Riemann tensor of a connection with skew torsion is our main motivation for the following definition, recalling also that in standard Riemannian geometry, a manifold (M, g) is said to be Einstein if $Z^g = 0$.

Definition 4.1. *Given an oriented Riemannian four-manifold (M, g, H) , we say that g is an Einstein metric with skew torsion, if*

$$Z^\nabla + S(\nabla * H) + * \frac{dH}{4} g = 0$$

where ∇ is the metric connection with skew torsion H .

We remark that the standard notion of Einstein metric is equivalent to having the induced Levi-Civita connections on Λ_+ and Λ_- self-dual and anti-self-dual, respectively. Our definition of Einstein with skew torsion simply adapts this, but we usually do not have both statements in our situation. Here we have chosen that ∇ on Λ_+ be self-dual. We see will later, in corollary 4.5, that for a compact manifold this choice does not constitute a problem.

Example 4.2. The very basic example is the one of the Lie group $S^1 \times S^3$, with one of the two flat connections given by left or right trivialization of the tangent bundle.

Example 4.3. Recall that the equations of type II string theory may be geometrically described as a tuple (M, g, H, ϕ, ψ) consisting of a manifold M with a Riemannian metric g , a three-form H , a so-called dilaton function ϕ and a spinor field ψ satisfying the following system of equations, [1],

$$\begin{aligned} \text{Ric}^\nabla + \frac{1}{2}d^*H + 2\nabla^g d\phi &= 0 & (\nabla_X^g + \frac{1}{4}X \lrcorner H)\psi &= 0 \\ d^*(e^{-2\phi}H) &= 0 & (2d\phi - H)\psi &= 0 \end{aligned}$$

where $\nabla = \nabla^g + \frac{1}{2}H$. Suppose $2d\phi = *H$, then the first equation implies

$$S(\text{Ric}^\nabla) + \nabla * H = 0$$

since $\nabla * H$ is the Hessian of 2ϕ and is therefore symmetric. Hence the trace-free part satisfies definition 4.1.

We have an interesting property in the compact case if the torsion is closed.

Proposition 4.4. *If M is compact and $dH = 0$, the Einstein equations with skew torsion imply that the vector field X defined by $i_X \omega_g = H$, where ω_g is the volume form, is a Killing field.*

Proof — It suffices to prove that

$$\int_M \|S(\nabla^g h)\|^2 \omega_g = 0,$$

where h is the one-form dual to X . We can write $S(\nabla^g h)$ as $\nabla^g h - \frac{1}{2}dh$ and using the Einstein condition with skew torsion also as $-(Z^g - \frac{1}{8}\|h\|^2 g + \frac{1}{2}h \otimes h)$. Observing that the decomposition $T^* \otimes T^* = S^2 T^* \oplus \Lambda^2 T^*$ is orthogonal we get that

$$\int_M \|S(\nabla^g h)\|^2 \omega_g = \int_M - \left(Z^g - \frac{1}{8}\|h\|^2 g + \frac{1}{2}h \otimes h, \nabla^g h \right) \omega_g$$

where the round brackets denote here the inner product of tensors, for convenience. It is easy to see that $(g, \nabla^g h) = 0$, since $d^*h = 0$; therefore we are left with

$$\int_M - \left(\text{Ric}^g + \frac{1}{2} h \otimes h, \nabla^g h \right) \omega_g.$$

Recall that the divergence of a two-symmetric tensor is given by $(\nabla^g)^*$, the formal adjoint of ∇^g . Recall also that by contracting the differential Bianchi identity, we get that the divergence of Ric^g is $-\frac{1}{2} ds^g$. Then we can write the integral as

$$\int_M \left(\frac{1}{2} ds^g - \frac{1}{2} \nabla^{g*}(h \otimes h), h \right) \omega_g.$$

The idea now is to write this as a divergence; since $d^*h = 0$, $(ds^g, h) = -\nabla^{g*}(s^g h)$ and $(\nabla^{g*}(h \otimes h), h) = \frac{1}{2} \nabla^{g*}(|h|^2 h)$. Finally we obtain that

$$\int_M \nabla^{g*} \left(-\frac{1}{2} s h + \frac{1}{4} |h|^2 h \right) \omega_g = 0.$$

□

The above result also derives from another interpretation of the Einstein equations with skew torsion which is related to conformal invariance and was originally proved in this context, [23].

Corollary 4.5. *On a compact four-manifold, an Einstein metric with closed skew torsion H satisfies the equation $Z^\nabla = 0$, where Z^∇ is the trace-free part of the Ricci tensor.*

Proof — Recall that a vector field X is said to be Killing if the symmetric part of $\nabla^g X$ vanishes. If X is the metric dual of h , where $h = *H$, then

$$\nabla^g X = \nabla^g h$$

and by means of Lemma 3.3

$$\nabla^g h = \nabla h.$$

Using proposition 4.4, then definition 4.1 gives $Z^\nabla = 0$.

□

Remark 4.6. It is clear, using the corollary above and looking at the expression of

$$Z^\nabla = Z^g + \frac{1}{2} * H \otimes * H - \frac{1}{8} \|H\|^2 g,$$

that if (M, g, H) is a compact Einstein manifold with closed skew torsion then so is $(M, g, -H)$.

4.1. An inequality. Our definition of Einstein metric with skew torsion implies that Λ_+ has a self-dual connection. This means that $\text{Tr}(R \wedge R) = f \omega_g$, where f is a non-negative function, and hence the first Pontryagin class of Λ_+ is non-negative. This implies a topological constraint on a compact four-manifold that generalizes the Hitchin-Thorpe inequality, [3, 16], which states that if M is a compact oriented Einstein manifold of dimension 4, then the Euler characteristic $\chi(M)$ and the signature $\tau(M)$ satisfy the inequality $\chi(M) \geq \frac{3}{2} |\tau(M)|$.

We have a similar result for connections with skew torsion.

Theorem 4.7. *Let (M, g, H) be a compact, oriented, four-dimensional Riemannian manifold, equipped with a metric connection with skew-symmetric torsion H , such that $Z^\nabla + S(\nabla *H + \frac{*dH}{4}g) = 0$, then*

$$\chi(M) \geq \frac{3}{2} |\tau(M)|.$$

Proof — We use the formulas discussed and proved in [4]. Both the Euler characteristic and the signature can be written in terms of the curvature operator \mathcal{R} as

$$\chi(M) = \frac{1}{8\pi^2} \int \text{Tr}(*\mathcal{R} * \mathcal{R}) \omega_g$$

$$\tau(M) = \frac{1}{12\pi^2} \int \text{Tr}(\mathcal{R} * \mathcal{R}) \omega_g$$

where $*$ is the Hodge star operator and ω_g the volume form with respect to the metric and the chosen orientation. Recall from the proof of theorem 3.2 that \mathcal{R} is given in blocks by

$$\mathcal{R} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Clearly, we have that $*A = A$, $*D = -D$ and since $B = 0$ we get $\text{Tr}(*\mathcal{R} * \mathcal{R}) = \text{Tr}(A^2 + D^2)$ and $\text{Tr}(\mathcal{R} * \mathcal{R}) = \text{Tr}(A^2 - D^2)$. Observe now that

$$\begin{aligned} \text{Tr}(*\mathcal{R} * \mathcal{R}) &= \text{Tr}(A^2 + D^2) \geq \text{Tr}(A^2 - D^2) = \text{Tr}(\mathcal{R} * \mathcal{R}) \\ \text{Tr}(*\mathcal{R} * \mathcal{R}) &= \text{Tr}(A^2 + D^2) \geq \text{Tr}(D^2 - A^2) = -\text{Tr}(\mathcal{R} * \mathcal{R}) \end{aligned}$$

which gives two inequalities $\chi(M) \geq \frac{3}{2}\tau(M)$ and $\chi(M) \geq -\frac{3}{2}\tau(M)$, and combining these two we get the desired inequality. \square

Our next goal is to determine for which Einstein metrics with skew torsion the equality is attained. First we need to look at the conformal class of such a metric.

4.2. Conformal invariance. We now introduce the notions of Weyl structure and Einstein-Weyl manifold, [8].

Definition 4.8. *Let M be a manifold with conformal structure $[g]$, i.e., an equivalence class of metrics such that $\tilde{g} \simeq g$ if $\tilde{g} = e^f g$, where $f : M \rightarrow \mathbb{R}$ is a smooth function. A Weyl connection is a torsion-free affine connection D such that for any representative of the metric g there exists a one-form ω such that $Dg = \omega \otimes g$. A Weyl manifold is a manifold equipped with a conformal structure and a compatible Weyl connection. The Weyl structure is said to be closed (resp. exact) if (any) ω is closed (resp. exact).*

We note that the notions of closed and exact Weyl structures are well defined. If ω is the one-form associated to g and $\tilde{\omega}$ is the one-form associated to $\tilde{g} = e^f g$, then $\tilde{\omega} = \omega + df$.

Definition 4.9. *A Weyl manifold is said to be Einstein-Weyl if the trace-free symmetric part of the Ricci tensor $S_0(\text{Ric}^D)$ vanishes.*

The following formulas, [21], are simple but extremely useful calculations:

The Weyl connection D with one-form ω is given explicitly by

$$(4.1) \quad D = \nabla_X^g Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X, Y)\omega^\sharp$$

where ω^\sharp denotes the vector field dual to ω . The symmetric part of its Ricci tensor is equal to

$$(4.2) \quad S(\text{Ric}^D) = \text{Ric}^g - \frac{1}{2}(\|\omega\|^2 g - \omega \otimes \omega) + S(\nabla^g \omega) - \frac{1}{2}(d^* \omega)g.$$

This immediately yields,

Theorem 4.10. *Let (M, g, H) be a four-dimensional Einstein manifold with skew torsion. Then if $\omega = *H$, the torsion-free connection D such that $D\omega = \omega \otimes g$ is an Einstein-Weyl connection. Conversely, given an Einstein-Weyl manifold, each metric in the conformal class defines, with $H = -*\omega$, an Einstein manifold with skew torsion.*

Proof — Suppose $(M, [g])$ is Einstein-Weyl. Take a representative of the metric g and its associated one-form ω . The connection defined by equation 4.1 has scalar curvature

$$s^D = s^g - \frac{3}{2}\|\omega\|^2 - 3d^* \omega.$$

Therefore, using also equation 4.2, the trace-free symmetric Ricci tensor is equal to

$$S_0(\text{Ric}^D) = \text{Ric}^g + \frac{1}{2}\omega \otimes \omega - \frac{1}{8}\|\omega\|^2 g + S(\nabla^g \omega) + \frac{1}{4}(d^* \omega)g.$$

Now take the metric connection with skew torsion $H = -*\omega$. Then clearly (M, g, H) is Einstein with skew torsion. The converse is perfectly analogous. \square

As an immediate corollary of this, we get that the Einstein equations with skew torsion are conformally invariant, that is, if the metric g is Einstein with skew torsion, then so are all metrics in the conformal class of g , if we transform the torsion appropriately.

Notice again that, unlike in string theory and Einstein-Weyl geometry, definition 4.1 does not work in any dimension except four. Indeed, it is crucial that $*H$ is a one-form.

Still in the context of conformal invariance we have the following important fact: given a metric g on a compact manifold and a one-form ω , there is a unique (up to a constant) metric $\tilde{g} = e^f g$ for some smooth function f , such that the one-form $\tilde{\omega} = \omega + df$ is co-closed with respect to \tilde{g} . This metric is of particular importance in Hermitian geometry and it is known in the literature as the Gauduchon gauge, [13]. We, then, have the following,

Corollary 4.11. *If (M, g, H) is a compact Einstein manifold with skew torsion then there exists a function f on M such that $(M, e^f g, e^f(H - *df))$ is Einstein with closed skew torsion.*

The above corollary together with corollary 4.5 implies that our definition of Einstein metrics with skew torsion is independent of orientation in the case of compact manifolds.

It should also be mentioned here that a generalization of the Hitchin-Thorpe inequality for Einstein-Weyl manifolds was proved in [20].

4.3. The equality. As mentioned in subsection 4.1 we are interested in the case where equality is achieved. The usual Riemannian situation was studied by N. Hitchin [3, 16], who proved that if M is a compact oriented four-dimensional Einstein manifold and the Euler characteristic $\chi(M)$ and the signature $\tau(M)$ satisfy

$$\chi(M) = \frac{3}{2}|\tau(M)|$$

then the Ricci curvature vanishes, and M is either flat or its universal cover is a $K3$ surface. In that case, M is either a $K3$ surface itself ($\pi_1(M) = 1$), or an Enriques surface ($\pi_1(M) = \mathbb{Z}_2$), or the quotient of an Enriques surface by a free antiholomorphic involution ($\pi_1(M) = \mathbb{Z}_2 \times \mathbb{Z}_2$) with the metric induced from a Calabi-Yau metric on $K3$.

In the following we investigate what happens when equality holds in our setting of connections with skew torsion. Given the link with Einstein-Weyl geometry, it is not surprising that a classification has been achieved for the four-dimensional case with closed Weyl structure, [14]. This is somewhat similar to what we want, but the arguments rely on twistor theory which we want to avoid, [14]. Instead we will keep to the language of Riemannian geometry.

Theorem 4.12. *Let (M, g, H) be a Riemannian compact, oriented four-manifold M which is an Einstein manifold with skew torsion satisfying the equality*

$$\chi(M) = \frac{3}{2}|\tau(M)|.$$

The either M is Einstein or its universal cover is isometric to $\mathbb{R} \times S^3$.

Remark 4.13. As mentioned before $M = S^1 \times S^3$ is a compact solution of the Einstein equations with skew torsion. Also observe that $S^1 \times S^3$ is not an Einstein manifold in the usual sense. Since $\chi(M) = 0$, if (M, g) was Einstein then we would have

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + \|W\|^2 \right) \omega_g$$

which would mean that both the scalar curvature and the Weyl tensor vanish. Therefore $S^1 \times S^3$ would be flat with respect to the Levi-Civita connection which is a contradiction.

Proof of theorem 4.12 — From corollary 4.11, we can assume that (M, g, H) is such that $dH = 0$. Let h be the torsion one-form. Suppose, without loss of generality, that $\chi(M) = -\frac{3}{2}\tau(M)$. Then $\text{Tr}(A^2) = 0$ (recall the proof of theorem 4.7) and from the decomposition of $\Lambda^2 \otimes \Lambda^2$ into irreducible $SO(4)$ -components, we get

$$\|W^+\|^2 = \|s^\nabla\|^2 = \|(d^*H)_+\|^2 = 0.$$

Then, in particular, $*dh$ is anti-self-dual, and we have

$$-\|dh\|^2 = \int_M *dh \wedge *dh = \int_M dh \wedge dh = \int_M d(h \wedge dh)$$

and so dh vanishes by Stokes theorem. Recall, from lemma 4.4, that if X is the dual of h via the metric g , then X is a Killing field. Combining these two facts we conclude that $\nabla^g X = 0$. Then either $X = 0$ and we are in the Einstein situation or otherwise X is a nowhere vanishing parallel vector field. In this case we have a reduction of the holonomy group and, by means of the de Rham decomposition theorem, M splits locally as a Riemannian product $\mathbb{R} \times N$. Since $\text{Ric}^\nabla = 0$ then

$$\text{Ric}^g = \frac{1}{2}\|h\|^2 g - \frac{1}{2}h \otimes h.$$

Observing that TN is the orthogonal complement of $\{X\}$, we conclude that N is Einstein with positive Ricci curvature. Hence, since N is of dimension 3, it is of positive sectional curvature. Therefore M is locally isometric to $\mathbb{R} \times S^3$, the metric splits as a product and the three-form is the pull-back of a three form in N , using the inclusion. \square

Remark 4.14. A natural question to ask is which compact Hermitian four-manifolds equipped with the Bismut are Einstein in the sense of our definition 4.1. The answer does not give new examples of such manifolds. We can prove that the Lee form is parallel and then repeat the steps of the proof above, [12].

Given what was presented here so far, a natural question to ask is if there are other instances of compact Einstein manifolds besides manifolds of type $S^1 \times S^3$. The answer is yes, and given theorem 4.10, a good source of examples is that of Einstein-Weyl geometry.

We can find a classification of four-dimensional Einstein-Weyl manifolds whose symmetry group is at least four dimensional in [19]. This article has two errors in the case of $U(2)$ -invariant structures which were pointed out by G. Bonneau in [6] who also offers a simpler description of the metrics in the Gauduchon gauge. We can summarize the results for the compact orientable case as follows:

If (M, g, H) is a compact orientable four-dimensional manifold which is Einstein with closed skew torsion and whose symmetry group is at least four-dimensional, then we have one of the following possibilities:

- if $*H$ is exact then M is Einstein,
- if $*H$ is closed but not exact then M is finitely covered by $S^1 \times S^3$ with its standard flat structure,
- if $*H$ is not closed then the symmetry group is
 - $S^1 \times SO(3)$ in which case M is S^4 , $S^1 \times S^3$, $S^1 \times_{(-1,-1)} S^3$, $S^2 \times S^2$ or $S^2 \times_{(-1,-1)} S^2$,
 - $U(2)$ in which case M is S^4 , $\mathbb{C}P^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$.

Also, for each of the listed manifolds there is, in fact, an Einstein structure with skew torsion.

5. INSTANTONS

As remarked in section 4, if we have an Einstein manifold with skew torsion (M, g, H) and ∇ is the metric connection with torsion H then the induced connection on Λ^+ is self-dual. In the particular case where M is a spin manifold, the induced connection on \mathcal{S}^+ , the bundle of positive half-spinors, is also self-dual. Self-dual connections are also called instantons.

By remark 4.6, in the compact case with closed H , we will have two different instantons. A question that arises here is whether or not such two instantons are always gauge equivalent. Note that if ∇^+ and ∇^- represent the two induced connection with torsion H and $-H$ respectively, the Yang-Mills density is the same, namely

$$|F^{\nabla^\pm}| = \left(|W^+|^2 + \left| \frac{s^{\nabla^\pm}}{12} \text{Id} \right|^2 + \left| \frac{(d^*H)_+}{2} \right|^2 \right) \text{dvol}.$$

We will now point out certain features of these concepts for a particular example.

5.1. An example on S^4 . Consider the family of $U(2)$ -invariant metrics on S^4 , mentioned in subsection 4.3. This is a one parameter family of Einstein metrics with skew torsion, presented explicitly in [6] in diagonal form by

$$(5.1) \quad ds^2 = \frac{2}{\Gamma} \left[\frac{k-x}{\Omega^2(x)(1+x^2)^2} (dx)^2 + \frac{k-x}{1+x^2} [(\sigma^1)^2 + (\sigma^2)^2] + \frac{\Omega^2(x)}{k-x} (\sigma^3)^2 \right]$$

$$(5.2) \quad H = 2 \frac{k-x}{(1+x^2)^2} dx \wedge \sigma^1 \wedge \sigma^2$$

where $x \in (-\infty, k)$ is a coordinate, $\{\sigma^i, i = 1, 2, 3\}$ is a basis of left-invariant forms such that $d\sigma^i = \frac{1}{2}\epsilon_{ijk}\sigma^j \wedge \sigma^k$,

$$\Omega^2(x) = 1 + n(x^2 - 1 - 2kx) \left(\frac{\pi}{2} + \arctan(x) \right) + n(x - 2k),$$

Γ is a positive homothetic parameter, k is a free parameter and n is such that

$$n = \frac{1}{k + (1 + k^2) \left(\frac{\pi}{2} + \arctan(k) \right)}.$$

Since Γ is simply a homothetic parameter, we can take it to be $\Gamma = 2$, for simplicity of calculations. For ease of notation, we will be writing

$$ds^2 = a^2 dx^2 + b^2 [(\sigma^1)^2 + (\sigma^2)^2] + c^2 (\sigma^3)^2$$

where

$$a^2 = \frac{k-x}{\Omega^2(x)(1+x^2)^2}, \quad b^2 = \frac{k-x}{1+x^2}, \quad c^2 = \frac{\Omega^2(x)}{k-x}.$$

We are interested in studying charge 1 instantons for (S^4, ds^2) under the gauge group $SU(2)$. Recall that such a moduli space for a round metric, i.e. a metric of constant sectional curvature, has been studied intensively and is well understood, [2, 11]. It is a five dimensional manifold which is diffeomorphic to hyperbolic space \mathbb{H}^5 . A natural question for us is whether the moduli space for each of the Bonneau metrics is smooth and if so what type of manifold is it? The answer to these queries can be obtained with the help of the following theorem.

Theorem 5.1 (Buchdahl, [7]). *Let X be a compact complex surface biholomorphic to a blow-up of \mathbb{CP}^2 n times, and $L_\infty \subset X$ be a rational curve with self-intersection $+1$. Let Y be a smooth four-manifold diffeomorphic to $n\mathbb{CP}^2$ obtained by collapsing L_∞ to a point y_∞ and reversing the orientation, and let $\pi : X \rightarrow Y$ be the collapsing map. If g is any smooth metric on Y such that π^*g is compatible with the complex structure on X , then there is a one-to-one correspondence between*

- (1) *equivalence classes of g -self-dual Yang-Mills connections on a unitary bundle E_{top} over Y , and*
- (2) *equivalence classes of holomorphic bundles E on X topologically isomorphic to π^*E whose restriction to L_∞ is holomorphically trivial and is equipped with a compatible unitary structure.*

For the case of $Y = S^4$, i.e. when $n = 0$, then $X = \mathbb{CP}^2$, and L_∞ can be taken to be a line in \mathbb{CP}^2 .

In view of this result, we will take the necessary steps to establish that the moduli space of instantons for S^4 with a Bonneau metric is smooth and moreover diffeomorphic to the one for S^4 with a round metric.

Take the round metric given by

$$g = \frac{dr^2 + r^2 + r^2[(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2]}{(1 + r^2)^2}$$

where r is a radial coordinate with $r \in (0, +\infty)$. A compatible almost complex structure is given by (1,0)-forms spanned by

$$\begin{aligned}\eta^1 &= dr + ir\sigma^3 \\ \eta^2 &= \sigma^1 + i\sigma^2\end{aligned}$$

This almost complex structure extends over to $r = 0$ and will be denoted by J_r .

Now, let us consider the Bonneau metrics. Here a compatible almost complex structure on $S^4 \setminus \{0, \infty\}$ is the one given by taking the (1,0)-forms to be spanned by

$$\begin{aligned}\theta^1 &= adx + ic\sigma^3 \\ \theta^2 &= \sigma^1 + i\sigma^2\end{aligned}$$

As we will see later this structure extends for $x = -\infty$. For now let us check that this is actually integrable on $S^4 \setminus \{\infty\}$. We have

$$\begin{aligned}d\theta^1 &= i\frac{c'}{a}\theta^1 \wedge \sigma^3 + ic\theta^2 \wedge \sigma^2 \\ d\theta^2 &= -i\theta^2 \wedge \sigma^3\end{aligned}$$

so both $d\theta^1$ and $d\theta^2$ are in the ideal generated by $\{\theta^1, \theta^2\}$. Call this complex structure J_B .

We wish to construct a diffeomorphism of S^4 such that it sends one almost complex structure into the other. It suffices to find a coordinate $R \in (0, +\infty)$ such that

$$f(dR + iR\sigma^3) = adx + ic\sigma^3$$

for some smooth function f . We have that R satisfies the following

$$\begin{cases} f dR = adx \\ f R = c \end{cases}$$

Then

$$\frac{dR}{R} = \frac{a}{c} dx \quad \Rightarrow \quad \log(R) = \int \frac{a}{c} dx$$

We now wish to show that this extends smoothly at $x = k$ and $x = -\infty$. Calculating the asymptotic expansion around $x = k$ for $\frac{a}{c}$, we have

$$\frac{a}{c} = (k - x)^{-1} + O(k - x).$$

Then

$$\frac{a}{c} \sim (k - x)^{-1} \quad \Rightarrow \quad \log(R) \sim -\log(k - x) \quad \Rightarrow \quad R \sim \frac{1}{k - x}.$$

For $x = -\infty$, we have

$$\frac{a}{c} = -x^{-1} + O(x^{-2})$$

and so near $-\infty$, $R \sim \frac{1}{x}$. In particular the complex structure compatible with the Bonneau metrics extends to $S^4 \setminus \{\infty\}$. We have then a diffeomorphism

$$\varphi : (S^4 \setminus \{\infty\}, J_r) \longrightarrow (S^4 \setminus \{\infty\}, J_B).$$

We consider the twistor space Z to S^4 with a Bonneau metric. The complex structure defined above on $S^4 \setminus \{\infty\}$ is compatible with the metric and gives a section of

$$Z \longrightarrow S^4 \setminus \{\infty\}.$$

On the other hand, the diffeomorphism φ identifies this with the complex structure of $\mathbb{CP}^2 \setminus \mathbb{CP}^1$. The fact that the diffeomorphism $\varphi : S^4 \rightarrow S^4$ commutes with the $U(2)$ -action means that

$$D\varphi_\infty : T_\infty \rightarrow T_\infty$$

is conformal (given by multiplication by a scalar) and so the \mathbb{CP}^1 over ∞ is sent to $L_\infty \subset X$ and the almost complex structures correspond. We can construct the diagram

$$\begin{array}{ccc} \mathbb{CP}^2 & \longrightarrow & X \\ \pi \downarrow & & \downarrow \bar{\pi} \\ S^4 & \xrightarrow{\varphi} & S^4 \end{array}$$

where X is biholomorphic to \mathbb{CP}^2 . Thus $\bar{\pi}^*(g)$, where g is a Bonneau metric, is compatible with the complex structure on X .

Remark 5.2. Note that the mapping $r \mapsto \frac{1}{r}$ provides the same result for a complex structure with the opposite orientation.

We have therefore checked all the conditions of theorem 5.1 and hence we deduce:

Theorem 5.3. *Let \mathcal{M}_B be the moduli space of $SU(2)$ -self-dual connections of charge 1 for a Bonneau metric on S^4 . Then \mathcal{M}_B is diffeomorphic to \mathcal{M} , the moduli space of $SU(2)$ -self-dual connections of charge 1 for a round metric on S^4 .*

If we consider the characterization of this moduli space in terms of $\mathbb{H}^5 = \mathbb{R}^4 \times \mathbb{R}^+ = \mathbb{C}^2 \times \mathbb{R}^+$ and identify \mathcal{M}_B with \mathcal{M} , then the $U(2)$ -invariant instanton equivalence classes are given by the curve

$$\{(z_1, z_2, t) : z_1 = z_2 = 0\}$$

and this contains the equivalence classes of the connections ∇^+ and ∇^- , the connections with skew torsion H and $-H$ respectively.

An interesting question is whether or not ∇^+ and ∇^- define the same point on this line, i.e. are gauge equivalent. The answer is no. We need only a counter-example so for simplicity we can choose the parameter k in (5.1) – (5.2) to be zero. We can then proceed to a somewhat lengthy calculation which goes as follows: suppose that there is a $SU(2)$ -gauge transformation $g : \mathcal{S}^+ \rightarrow \mathcal{S}^+$ such that $g^{-1}\nabla^+g = \nabla^-$. Then g is a section of $\mathcal{S}^+ \otimes \mathcal{S}^+$ which is covariantly constant under the tensor product connection $\nabla = \nabla^+ \otimes 1 + 1 \otimes \nabla^-$. In this case, g will be annihilated by the curvature of ∇ , R^∇ . We can check that there is only one g such that $R^\nabla g = 0$ and that it is totally determined by the metric ds^2 and the three form H extending both at $x = k$ and $x = -\infty$. We can then compute ∇g and see that g is not parallel.

Remark 5.4. Using the reduction procedure of Cavalcanti, [10], the closed three form H will induce a closed three form on \mathcal{M}_B , giving it the structure of a manifold with skew torsion.

Acknowledgments: The author would like to thank Nigel Hitchin for pointing her to this topic, and for many helpful conversations that have ensued. This research was partially supported by the Portuguese Foundation for Science and Technology (FCT) through the POPH-QREN scholarship program and by the Research Center of Mathematics of the University of Minho through the FCT pluriannual funding program.

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